FORM TP 2008240

TEST CODE **22134020**

MAY/JUNE 2008

CARIBBEAN EXAMINATIONS COUNCIL

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 1 – PAPER 02

ALGEBRA, GEOMETRY AND CALCULUS

2 ½ hours

30 JUNE 2008 (a.m.)

This examination paper consists of **THREE** sections: Module 1, Module 2 and Module 3.

Each section consists of 2 questions.
The maximum mark for each Module is 50.
The maximum mark for this examination is 150.
This examination consists of 6 printed pages.

INSTRUCTIONS TO CANDIDATES

1. **DO NOT** open this examination paper until instructed to do so.
2. Answer **ALL** questions from the **THREE** sections.
3. Write your solutions, with full working, in the answer booklet provided.
4. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to three significant figures.

Examination Materials Permitted

Graph paper (provided)
Mathematical formulae and tables (provided) – **Revised 2008**
Mathematical instruments
Silent, non-programmable, electronic calculator

4020

SECTION A (Module 1)

Answer BOTH questions.

1. (a) (i) Determine the values of the real number h for which the roots of the quadratic equation $4x^2 - 2hx + (8 - h) = 0$ are real. **[8 marks]**

(ii) The roots of the cubic equation

$$x^3 - 15x^2 + px - 105 = 0$$

are $5 - k$, 5 and $5 + k$.

Find the values of the constants p and k. **[7 marks]**

(b) (i) Copy the table below and complete by inserting the values for the functions $f(x) = |\,x + 2\,|$ and $g(x) = 2\,|\,x - 1\,|$.

x	-3	-2	-1	0	1	2	3	4	5
$f(x)$	1		1		3			6	
$g(x)$	8	6		2		2			

[4 marks]

(ii) Using a scale of 1 cm to 1 unit on **both** axes, draw on the **same** graph

$f(x)$ **and** $g(x)$ for $-3 \leq x \leq 5$. **[4 marks]**

(iii) Using the graphs, find the values of x for which $f(x) = g(x)$. **[2 marks]**

Total 25 marks

GO ON TO THE NEXT PAGE

2. (a) Without using calculators or tables, evaluate

$$\sqrt{\frac{27^{10} + 9^{10}}{27^4 + 9^{11}}}.$$

[8 marks]

(b) (i) Prove that $\log_n m = \dfrac{\log_{10} m}{\log_{10} n}$, for $m, n \in \mathbf{N}$. [4 marks]

(ii) Hence, given that $y = (\log_2 3)(\log_3 4)(\log_4 5) \ldots (\log_{31} 32)$, calculate the **exact** value of y. [6 marks]

(c) Prove, by the principle of mathematical induction, that

$$f(n) = 7^n - 1$$

is divisible by 6, for all $n \in \mathbf{N}$. [7 marks]

Total 25 marks

SECTION B (Module 2)

Answer BOTH questions.

3. (a) Let $\mathbf{p} = \mathbf{i} - \mathbf{j}$. If $\mathbf{q} = \lambda\mathbf{i} + 2\mathbf{j}$, find values of λ such that

(i) $\mathbf{q}$ is parallel to $\mathbf{p}$ [1 mark]

(ii) $\mathbf{q}$ is perpendicular to $\mathbf{p}$ [2 marks]

(iii) the angle between $\mathbf{p}$ and $\mathbf{q}$ is $\dfrac{\pi}{3}$. [5 marks]

(b) Show that $\dfrac{1 - \cos 2A + \sin 2A}{1 + \cos 2A + \sin 2A} = \tan A$. [6 marks]

(c) (i) Using the formula for $\sin A + \sin B$, show that if $t = 2\cos\theta$ then

$$\sin(n+1)\,\theta = t \sin n\theta - \sin(n-1)\,\theta$$ [2 marks]

(ii) Hence, show that $\sin 3\theta = (t^2 - 1)\sin\theta$. [2 marks]

(iii) Using (c) (ii) above, or otherwise, find ALL solutions of $\sin 3\theta = \sin\theta$, $0 \leq \theta \leq \pi$. [7 marks]

Total 25 marks

GO ON TO THE NEXT PAGE

4. (a) (i) The line $x - 2y + 4 = 0$ cuts the circle, $x^2 + y^2 - 2x - 20y + 51 = 0$ with centre P, at the points A and B.

Find the coordinates of P, A and B. [6 marks]

(ii) The equation of any circle through A and B is of the form

$$x^2 + y^2 - 2x - 20y + 51 + \lambda\,(x - 2y + 4) = 0$$

where λ is a parameter.

A new circle C with centre Q passes through P, A and B.

Find

a) the value of λ [2 marks]

b) the equation of circle C [2 marks]

c) the distance, $|\,PQ\,|$, between the centres [3 marks]

d) the distance $|\,PM\,|$ if PQ cuts AB at M. [4 marks]

(b) A curve is given by the parametric equations $x = 2 + 3\sin t$, $y = 3 + 4\cos t$.

Show that

(i) the Cartesian equation of the curve is

$$\frac{(x-2)^2}{9} + \frac{(y-3)^2}{16} = 1$$ [3 marks]

(ii) every point on the curve lies within or on the circle

$$(x-2)^2 + (y-3)^2 = 25.$$ [5 marks]

Total 25 marks

GO ON TO THE NEXT PAGE

SECTION C (Module 3)

Answer BOTH questions.

5. (a) Use L'Hopital's rule to obtain $\lim_{x \to 0} \frac{\sin 4x}{\sin 5x}$. **[3 marks]**

(b) (i) Given that $y = \frac{x}{1 - 4x}$,

a) find $\frac{dy}{dx}$ **[4 marks]**

b) show that $x^2 \frac{dy}{dx} = y^2$. **[2 marks]**

(ii) Hence, or otherwise, show that $x^2 \frac{d^2y}{dx^2} + 2\,(x - y)\,\frac{dy}{dx} = 0$. **[3 marks]**

(c) A rectangular box **without a lid** is made from thin cardboard. The sides of the base are $2x$ cm and $3x$ cm, and its height is h cm. The total surface area of the box is 200 cm^2.

(i) Show that $h = \frac{20}{x} - \frac{3x}{5}$. **[4 marks]**

(ii) Find the height of the box for which its volume V cm^3 is a maximum.

[9 marks]

Total 25 marks

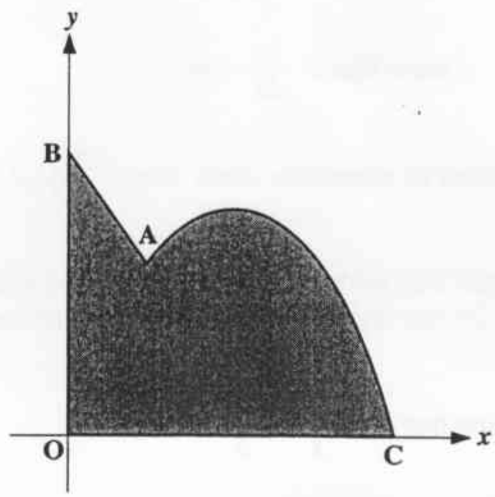
GO ON TO THE NEXT PAGE

6. (a) Use the substitution $u = 3x^2 + 1$ to find $\int \frac{x\,dx}{\sqrt{3x^2 + 1}}$. **[6 marks]**

(b) A curve C passes through the point (3, –1) and has gradient $x^2 - 4x + 3$ at the point (x, y) on C.

Find the equation of C. **[4 marks]**

(c) The figure below **(not drawn to scale)** shows part of the line $y + 2x = 5$ and part of the curve $y = x(4 - x)$ which meet at A. The line meets Oy at B and the curve cuts Ox at C.



(i) Find the coordinates of A, B and C. **[6 marks]**

(ii) Hence find the **exact** value of the area of the shaded region. **[9 marks]**

Total 25 marks

END OF TEST